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PASSING FROM MECHANICS OF THE SPECIAL THEORY OF RELATIVITY TO NEWTONIAN MECHANICS, AND THE RELATIVISTIC EFFECTS*

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A general method of passing from mechanics of the special theory of relativity to Newtonian mechanics is considered together with various relativistic effects.

1. The basic variational equation. Let x^k be the observer's coordinate system in a Minkowski space and $g_{ij}(x^k)$ be components of the metric tensor of that system. Below, the Latin and Greek indices assume, respectively, the values 1, 2, 3, 4, and 1, 2, 3. In the system of coordinates x^k which corresponds to the observer's inertial synchronous reference system (inertial reference systems are assumed in what follows to be also synchronous) we have

$$x^4 = ct, \quad g_{ij} = \left\| \begin{array}{c:c} -h_{\alpha\beta} & 0 \\ \hdashline 0 & 1 \end{array} \right\| \tag{1.1}$$

where c is the speed of light in vacuum, t is the observer's time, and the quantities $h_{\alpha\beta}$ are generally dependent on space coordinates x^ν which in the special theory of relativity can always be chosen so that throughout the space we have $h_{\alpha\beta} = 1$ when $\alpha = \beta$ and $h_{\alpha\beta} = 0$ when $\alpha \neq \beta$. (In the general theory of relativity this can be achieved locally along any specified curve. Respective coordinate systems are called Fermi coordinates).

The definition of motion of a continuous medium is linked with the introduction, besides the observer's system of coordinates x^k, of the system of coordinates ξ^j accompanying the medium. The latter individualize with $j = 1, 2, 3$ infinitely small particles of the medium, and are called Lagrangian coordinates. The quantity ξ^4 is a parameter of points on world lines of the medium along which $\xi^\alpha = \text{const}$. The relation between coordinate x^k and ξ^j is defined by the law of motion of the medium

$$x^k = x^k(\xi^j) \tag{1.2}$$

The four-dimensional velocity $\mathbf{u}$ of the medium is defined by the unit vector tangent to the medium world lines. In the observer's coordinates its components are

$$u^k = \left(\frac{dx^k}{ds}\right)_{\xi^\nu} = \frac{dx^k}{c\,d\tau} = \left\| \frac{v^\alpha}{(c^2 - v^2)^{1/2}} \,\Big|\, \frac{c}{(c^2 - v^2)^{1/2}} \right\| \tag{1.3}$$

$$u_k = g_{ki}u^i = \left\| \frac{-v_\alpha}{(c^2 - v^2)^{1/2}} \,\Big|\, \frac{c}{(c^2 - v^2)^{1/2}} \right\|$$

$$ds = [g_{ij}(dx^i)_{\xi^\nu}(dx^j)_{\xi^\nu}]^{1/2} = (c^2 - v^2)^{1/2}(dt)_{\xi^\nu} \equiv c\,d\tau$$

$$v^\alpha = (dx^\alpha/dt)_{\xi^\nu}, \quad v^2 = h_{\alpha\beta}v^\alpha v^\beta, \quad v_\alpha = h_{\alpha\beta}v^\beta$$

where ds is an element of the medium world line, $d\tau$ is the increment of the proper time, and v^α are components of three-dimensional velocity in the observer's inertial reference system.

The design of continuous medium models is always associated with the acceptance of certain postulates. It is most rational to start from the basic variational equation /1–3/

$$\delta \frac{1}{c} \int\limits_{V_4} \Lambda\, dV_4 + \delta W^* + \delta W = 0 \tag{1.4}$$

$$dV_4 = \sqrt{-g}\, d^4x = \sqrt{-g^\wedge}\, d^4\xi = dV_3^\wedge\, ds, \quad -\Lambda\, dV_3^\wedge = dE$$

$$g = |g_{ij}|, \quad g^\wedge = |g_{ij}^\wedge|, \quad d^4x = dx^1 \ldots dx^4, \quad d^4\xi = d\xi^1 \ldots d\xi^4$$

where Λ is the Lagrangian of the continuous medium, δW^* is a specified functional that takes

*Prikl.Matem.Mekhan.,45,No.6,985-993,1981 into account irreversible processes and external actions, δW is a functional determined by Eq. (1.4) and representing a surface integral over the boundary of region V_4 in the Minkowski space and, also, possibly over the two sides of strong discontinuity surfaces inside V_4, $dV_3^{\wedge}$ is the three-dimensional volume of an individual infinitely small particle of continuous medium, defined in the accompanying coordinate system, and dE is the internal energy of the particle of volume $dV_3^{\wedge}$. Variations in Eq. (1.4) are taken with $\xi^j = \text{const}$. Quantities related to the accompanying coordinate system are denoted by the symbol $\wedge$.

Models of continuous media are fixed by the selection of the Lagrangian and the functional δW^*. For example, setting

$$\Lambda = -\rho c^2 - \rho U(S, \gamma_{ij}^{\wedge}, \overset{\circ}{\gamma}{}_{ij}^{\wedge}, K_B^{\wedge}) \tag{1.5}$$

$$\delta W^* = \frac{1}{c}\int_{V_4} (\rho T\delta S - Q_i \delta x^i)\, dV_4, \quad Q_i u^i = \frac{dQ^{(e)}}{c\, d\tau}$$

$$\left(\gamma_{ij}^{\wedge} = u_i^{\wedge} u_j^{\wedge} - g_{ij}^{\wedge}, \quad u_i^{\wedge} = \frac{g_{i4}^{\wedge}}{\sqrt{g_{44}^{\wedge}}} = g_{ij}^{\wedge}\frac{d\xi^j}{ds}, \; g_{ij}^{\wedge} = g_{kr} x_i^k x_j^r\right.$$

$$\left. x_i^k = \frac{\partial x^k}{\partial \xi^i}, \; \rho = \rho^\circ \sqrt{\frac{\gamma^\circ}{\gamma^\wedge}}, \; \gamma^\wedge = |\gamma_{\alpha\beta}^{\wedge}|, \; \gamma^\circ = |\overset{\circ}{\gamma}{}_{\alpha\beta}^{\wedge}| \right)$$

where all parameters $K_B^{\wedge}$ are known nonvaried functions of Lagrangian coordinates ξ^α, we obtain the model of a perfectly elastic body in which, by definition, the processes are reversible. The introduced arguments of the Lagrangian and of functional δW^* have the following physical meaning: ρ is the density of the mass of body at rest, S is the specific entropy of the medium, T is the absolute temperature, Q_i are density components of the 4-force, $dQ^{(e)}$ is the density of external heat volume influx to the medium during time $d\tau = ds/c$ (for more complex models of media the quantity $Q_i u^i$ may also include the external influx of energy other than heat), $g_{ij}^{\wedge}$, $u_i^{\wedge}$ are components of the metric tensor and of 4-velocity medium in the accompanying coordinate system, and $\overset{\wedge}{ij}$ are components of the space metric tensor, which satisfy the relations

$$u^i \gamma_{ij} = u^j \gamma_{ij} = 0, \quad \gamma_{4j}{}^* = \gamma_{i4}{}^* = 0, \quad \gamma_{4j}^{\wedge} = \gamma_{i4}^{\wedge} = 0 \tag{1.6}$$

and determine in the accompanying coordinate system the length dl of infinitely small segments of the continuous medium that correspond to infinitely small increments of Lagrangian coordinates $d\xi^\alpha$

$$dl^2 = \gamma_{\alpha\beta}^{\wedge}\, d\xi^\alpha\, d\xi^\beta = \gamma_{ij}^{\wedge}\, d\xi^i\, d\xi^j$$

ρ° is the density of the medium mass at rest in the "initial state", $\overset{\circ}{\gamma}{}_{ij}^{\wedge}$ are components of the space metric tensor in the initial state which satisfy the relations

$$u^i \gamma_{ij}{}^\circ = u^j \gamma_{ij}{}^\circ = 0, \quad \overset{\circ}{\gamma}{}_{4j}^{*} = \overset{\circ}{\gamma}{}_{i4}^{*} = 0, \quad \overset{\circ}{\gamma}{}_{4j}^{\wedge} = \overset{\circ}{\gamma}{}_{i4}^{\wedge} = 0 \tag{1.7}$$

and determine dl° in the initial state

$$dl^{\circ 2} = \overset{\circ}{\gamma}{}_{\alpha\beta}^{\wedge}\, d\xi^\alpha\, d\xi^\beta = \overset{\circ}{\gamma}{}_{ij}^{\wedge}\, d\xi^i\, d\xi^j \tag{1.8}$$

The quantities ρ°, $\overset{\circ}{\gamma}{}_{ij}^{\wedge}$ are by definition known nonvaried functions of ξ^α, i.e. belong to parameters of the $K_B^{\wedge}$ type. Quantities denoted by an asterisk relate to the reference system that is proper for the considered point ξ^α_* of the medium at the instant of time t_*. The so-called observer's inertial reference system for which $v^{*\nu}(\xi_*^\alpha, t_*) = 0$, hence $u^{*i}(\xi_*^\alpha, t_*) = \delta_4^i$. (In the Minkowski space some proper coordinate system corresponds to each proper reference system).

The medium density ρ defined by the fifth equality in parentheses in (1.5) satisfies the relativistic continuity equation

$$\nabla_i \rho u^i = 0$$

Formulas (1.6) and (1.7) imply that under the transformation

$$\xi'^\alpha = \xi'^\alpha(\xi^\beta), \; \xi'^4 = \xi'^4(\xi^\beta, \xi^4) \tag{1.9}$$

of the accompanying system of coordinates ξ^i, the quantities $\gamma_{\alpha\beta}^{\wedge}$, $\overset{\circ}{\gamma}{}_{\alpha\beta}^{\wedge}$ behave as covariant components of three-dimensional tensors, for instance

$$\gamma_{\alpha\beta}^{\wedge\prime} = \frac{\partial \xi^i}{\partial \xi'^\alpha}\frac{\partial \xi^j}{\partial \xi'^\beta}\gamma_{ij}^{\wedge} = \frac{\partial \xi^\delta}{\partial \xi'^\alpha}\frac{\partial \xi^\nu}{\partial \xi'^\beta}\gamma_{\delta\nu}^{\wedge}$$

Similar relations hold for $\overset{\circ}{\gamma}{}_{\alpha\beta}^{\wedge}$, and

$$\gamma_{i4}^{\wedge\prime} = \gamma_{4j}^{\wedge\prime} = 0, \quad \overset{\circ}{\gamma}_{i4}^{\wedge\prime} = \overset{\circ}{\gamma}_{4j}^{\wedge\prime} = 0$$

Because of this the ratio $\gamma^\circ/\gamma^\wedge$ and density ρ are invariant to transformations (1.9) of the accompanying coordinate system. Since the Lagrangian is by definition a four-dimensional scalar, hence function U in the expression for Λ in (1.5) must also be invariant to transformations (1.9). It follows from this that for an isotropic body, for which all $K_B^\wedge$ are scalars, function U can depend on components $\gamma_{ij}^\wedge$, $\overset{\circ}{\gamma}_{ij}^\wedge$ only in terms of such of their combinations that are themselves invariant to transformations (1.9). It can be shown using formulas (1.6) and (1.7) that the number of such independent combinations is three. For example, they can be represented by the scalars

$$\begin{aligned}
&I_1^\circ = \gamma^{\circ\wedge\alpha\beta}\varepsilon_{\alpha\beta}^\wedge \equiv \gamma^{\circ\wedge ij}\varepsilon_{ij}^\wedge \\
&I_2^\circ = \gamma^{\circ\wedge\alpha\delta}\gamma^{\circ\wedge\beta\gamma}\varepsilon_{\alpha\beta}^\wedge\varepsilon_{\gamma\delta}^\wedge \equiv \gamma^{\circ\wedge ik}\gamma^{\circ\wedge jr}\varepsilon_{ij}^\wedge\varepsilon_{rk}^\wedge \\
&I_3^\circ = \gamma^{\circ\wedge\alpha\zeta}\gamma^{\circ\wedge\beta\gamma}\gamma^{\circ\wedge\delta\varepsilon}\varepsilon_{\alpha\beta}^\wedge\varepsilon_{\gamma\delta}^\wedge\varepsilon_{\varepsilon\zeta}^\wedge \equiv \gamma^{\circ\wedge in}\gamma^{\circ\wedge jr}\gamma^{\circ\wedge km}\varepsilon_{ij}^\wedge\varepsilon_{rk}^\wedge\varepsilon_{mn}^\wedge \\
&\left(\varepsilon_{ij}^\wedge \equiv \frac{1}{2}(\gamma_{ij}^\wedge - \overset{\circ}{\gamma}_{ij}^\wedge), \ \gamma^{\circ\wedge\alpha\beta} \equiv \frac{1}{\gamma^\circ}\frac{\partial\gamma^\circ}{\partial\overset{\circ}{\gamma}_{\beta\alpha}^\wedge}\right)
\end{aligned}$$

The four-dimensional components $\gamma^{\circ\wedge i4} = \gamma^{\circ\wedge 4i}$ do not affect invariants I_α°, since $\varepsilon_{i4}^\wedge = \varepsilon_{4i}^\wedge = 0$. Moreover it is possible to assume for definiteness that in the fixed accompanying system of coordinates $\gamma^{\circ\wedge i4} = \gamma^{\circ\wedge 4i} = 0$. The quantities $\gamma^{\circ\wedge\alpha\beta}$ under transformations (1.9) of the accompanying system of coordinates ξ^i behave as contravariant components of the three-dimensional tensor

$$\gamma^{\circ\wedge\prime\alpha\beta} = \frac{\partial\xi'^\alpha}{\partial\xi^i}\frac{\partial\xi'^\beta}{\partial\xi^j}\gamma^{\circ\wedge ij} = \frac{\partial\xi'^\alpha}{\partial\xi^\delta}\frac{\partial\xi'^\beta}{\partial\xi^\gamma}\gamma^{\circ\wedge\delta\gamma}$$

2. Euler's equations and conditions at discontinuities.

We select functions $x^i(\xi^j)$, $S(\xi^j)$ as the independent determining parameters. On the basis of the above definitions of quantities $\gamma_{ij}^\wedge$, $\overset{\circ}{\gamma}_{ij}^\wedge$, ρ, $K_B^\wedge$, dV_4 and of the theory of variations developed in /3/ we have

$$\begin{aligned}
&\delta\gamma_{ij}^\wedge = -\gamma_{q(i}^\wedge\gamma_{j)p}^\wedge\xi_r{}^q\xi_k{}^p\nabla^k\delta x^r, \quad \delta\rho = \rho\gamma_{ij}\nabla^j\delta x^i \\
&\delta\, dV_4 = (\nabla_i\delta x^i)\, dV_4, \quad \delta\overset{\circ}{\gamma}_{ij}^\wedge = 0, \quad \delta K_B^\wedge = 0 \\
&\delta x^{\wedge q} = \xi_i{}^q\delta x^i, \quad \xi_i{}^q \equiv \partial\xi^q/\partial x^i
\end{aligned} \tag{2.1}$$

As the result of variations in Eq.(1.4) with allowance for formulas (2.1) we obtain the relation

$$\begin{aligned}
&\int_{V_4}\left[(\nabla^jT_{ij} - Q_i)\,\delta x^i + \rho\left(T - \frac{\partial U}{\partial S}\right)\delta S - \nabla^j(T_{ij}\delta x^i)\right] dV_4 + \delta W = 0 \\
&T_{ij} = \rho(c^2 + U)\,u_iu_j - p_{ij}, \quad p_{ij} = \xi_i{}^k\xi_j{}^r p_{kr}^\wedge, \quad p_{ij}^\wedge = 2\rho\frac{\partial U}{\partial\gamma_{\alpha\beta}^\wedge}\gamma_{\alpha i}^\wedge\gamma_{\beta j}^\wedge
\end{aligned} \tag{2.2}$$

From this, owing to the arbitrariness of variations δx^i, δS and the definition of functional δW, we obtain Euler's equations and the expression for the functional δW

$$\nabla^jT_{ij} = Q_i, \quad \frac{\partial U}{\partial S} = T, \quad \delta W = \frac{1}{c}\int_{\partial V_4 + \Sigma_\pm} T_{ij}\delta x^i n^j\, d\Sigma \tag{2.3}$$

where $\Sigma_\pm$ are the two sides of the three-dimensional discontinuity surface inside region V_4, $d\Sigma$ is an element of the three-dimensional volume of surface $\partial V_4 + \Sigma_\pm$, n^j are components of the 4-vector of the external normal to that surface. By defining functional δW for $\delta x^i = 0$ at the boundary ∂V_4 we can obtain conditions at the surface of strong discontinuity. For instance, in the absence of external actions we have $\delta W = 0$, hence for variations δx^i that are continuous at the discontinuity from (2.3) for the functional δW we obtain

$$(T_{ij})_+ n^j = (T_{ij})_- n^j, \quad n^j \equiv n_+{}^j = -n_-{}^j$$

In (2.3) the first equation is the equation of energy-momentum of an ideal gas, T_{ij} are components of the complete energy-momentum tensor and p_{ij} are components of the four-dimensional tensor of internal stresses. Equations (2.3) retain their form also in the accompanying coordinate system, except that the symbol $\wedge$ must be added to all tensor components.

The definition (2.2) of quantities $p_{ij}^{\wedge}$, p_{ij} implies that

$$p_{\alpha\beta}^{\wedge} = 2\rho \frac{\partial U}{\partial \gamma_{\gamma\delta}^{\wedge}} \gamma_{\gamma\alpha}^{\wedge} \gamma_{\delta\beta}^{\wedge}, \quad p_{i4}^{\wedge} = p_{4i}^{\wedge} = 0, \quad u^k p_{ik} = 0 \tag{2.4}$$

The substitution of expression (1.3) in the last of equalities (2.4) yields for u^k

$$p_{\alpha 4} = -\frac{v^\beta}{c} p_{\alpha\beta}, \quad p_{44} = -\frac{v^\beta}{c} p_{4\beta} = \frac{v^\alpha v^\beta}{c^2} p_{\alpha\beta} \tag{2.5}$$

In the observer's inertial reference system the first of Eqs.(2.3), with expression (2.2) for components T_{ij} taken into account, assumes the form

$$\frac{1}{c}\frac{\partial}{\partial t}[\rho(c^2+U)u_k u_4 - p_{k4}] - \nabla_{(h)}^{\beta}[\rho(c^2+U)u_k u_\beta - p_{k\beta}] = Q_k \tag{2.6}$$

where $\nabla_{(h)}^{\beta}$ is the contravariant component in the three-dimensional space with components of the metric tensor $h_{\alpha\beta}$. Substituting expressions (1.3) for u_k into Eq.(2.6) and expression (2.5) for p_{k4}, for the equations of momenta and the equation of energy, with $k=\alpha$ and $k=4$, we obtain

$$\frac{\partial}{\partial t}\left[\rho(c^2+U)\frac{v_\alpha}{c^2-v^2} - p_{\alpha\beta}\frac{v^\beta}{c^2}\right] + \nabla_{(h)}^{\beta}\left[\rho(c^2+U)\frac{v_\alpha v_\beta}{c^2-v^2} - p_{\alpha\beta}\right] = h_{\alpha\beta}Q^\beta \tag{2.7}$$

$$\frac{\partial}{\partial t}\left[\rho(c^2+U)\frac{c^2}{c^2-v^2} - p_{\alpha\beta}\frac{v^\alpha v^\beta}{c^2}\right] + \nabla_{(h)}^{\beta}\left[\rho(c^2+U)\frac{v_\beta c^2}{c^2-v^2} - p_{\alpha\beta}v^\alpha\right] = cQ_4$$

where in conformity with (2.2) and (2.4)

$$p_{\alpha\beta} = p_{ij}^{\wedge}\frac{\partial \xi^i}{\partial x^\alpha}\frac{\partial \xi^j}{\partial x^\beta} = p_{\gamma\delta}^{\wedge}\left(\frac{\partial \xi^\gamma}{\partial x^\alpha}\right)_t\left(\frac{\partial \xi^\delta}{\partial x^\beta}\right)_t, \quad p_{\gamma\delta}^{\wedge} = 2\rho\frac{\partial U}{\partial \gamma_{\alpha\beta}^{\wedge}}\gamma_{\alpha\gamma}^{\wedge}\gamma_{\beta\delta}^{\wedge}$$

Let us also write the continuity equation in the three-dimensional form

$$\frac{\partial}{\partial t}\frac{\rho}{(c^2-v^2)^{1/2}} + \nabla_\alpha\frac{\rho v^\alpha}{(c^2-v^2)^{1/2}} = 0 \tag{2.8}$$

If at a given instant of time the observer's reference system for an infinitely small particle of a continuous medium is its proper one, Eqs.(2.8) and (2.7) are substantially simplified:

$$\frac{\partial \rho}{\partial t^*} + \rho\nabla_{(h)\alpha}^{*}v^{*\alpha} = 0 \tag{2.9}$$

$$\left[\rho\left(1+\frac{U}{c^2}\right)h_{\alpha\beta}^{*} - \frac{1}{c^2}p_{\alpha\beta}^{*}\right]\frac{\partial v^{*\beta}}{\partial t^*} = \nabla_{(h)}^{*\beta}p_{\alpha\beta}^{*} + h_{\alpha\beta}^{*}Q^{*\beta} \tag{2.10}$$

$$\rho\frac{\partial U}{\partial t^*} = p_{\alpha\beta}^{*}\nabla_{(h)}^{*\beta}v^{*\alpha} + cQ_4^{*}, \quad cQ_4^{*} = cQ_i u^i = \frac{dQ^{(e)}}{d\tau} = \frac{dQ^{(e)}}{dt^*} \tag{2.11}$$

where

$$p_{\alpha\beta}^{*} = 2\rho\frac{\partial U}{\partial \gamma_{\gamma\delta}^{\wedge}}\left(\frac{\partial x^{*\mu}}{\partial \xi^\gamma}\right)_{t^*}\left(\frac{\partial x^{*\nu}}{\partial \xi^\delta}\right)_{t^*}h_{\mu\alpha}^{*}h_{\nu\beta}^{*}, \quad p_{i4}^{*} = p_{4i}^{*} = 0 \tag{2.12}$$

$$\gamma_{\alpha\beta}^{\wedge} = h_{\gamma\delta}^{*}\left(\frac{\partial x^{*\gamma}}{\partial \xi^\alpha}\right)_{t^*}\left(\frac{\partial x^{*\delta}}{\partial \xi^\beta}\right)_{t^*}, \quad \gamma_{\alpha\beta}^{*} = h_{\alpha\beta}^{*}$$

$$T_{\alpha\beta}^{*} = -p_{\alpha\beta}^{*}, \quad T_{\alpha 4}^{*} = T_{4\alpha}^{*} = 0, \quad T_{44}^{*} = \rho(c^2+U)$$

The problem of establishing the laws of motion in arbitrarily specified observer's reference systems on the basis of known laws in the proper reference system is a problem of the theory of navigation. The passage from a proper reference system in which at the considered point $g_{ij}^{*} = g_{ij}$ to an arbitrary inertial reference system is achieved with the use of the Lorentz transformation. As the result, the relativistic equations of continuity and motion can be represented in any arbitrary observer's reference system in the form

$$\nabla_j(\rho d_i^{\ j} u^{*i}) = 0, \quad \nabla^j(b_i^{\ k} b_j^{\ r} T_{kr}^{*}) = b_i^{\ k} Q_k^{*}$$

where $d_i^{\,j}$ are components of the matrix of transformation that links the bases $э_j$ and $э_i^*$ of the observer's and of the indicated proper coordinate systems, respectively, as follows:

$$э_i^* = d_i^{\,j} э_j$$

and $b_i^{\,k}$ are components of matrix inverse of $\| d_i^{\,j} \|$. When the bases $э_i^*$, $э_j$ are orthonormalized and their space vectors $э_\alpha^*$, $э_\beta$ are identically oriented in a three-dimensional physical space, components $d_i^{\,j}$ are of the form indicated in /4/.

3. Passing to Newtonian mechanics. Equations (2.9)—(2.11) differ from equations of the model of an ideal elastic body in Newtonian mechanics written in a proper reference system only by the terms with coefficients c^{-2}. If $U=0$ (this equality holds, e.g., for dust), then $p_{\alpha\beta}=0$ and the indicated terms are absent. In such case equations of relativistic and Newtonian mechanics in the proper reference system are the same, but in arbitrary inertial reference systems they differ, since the first are the result of navigational recalculation using the Lorentz transform, while the second are obtained as the solution of the navigational problem using the Galilean transform (see the footnote on p. 734 in the preceding paper L.I. Tkachev). Equations of relativistic and Newtonian mechanics expressed in arbitrary inertial reference systems prove to be invariant to the Lorentz or Galilean transforms. If $U \neq 0$, Eqs.(2.9)—(2.11) with terms with coefficients c^{-2} can also be considered in Newtonian mechanics as laws expressed in the proper reference system, which define the motion of some ideally elastic body with complicated properties. In that case the respective laws of motion in an arbitrary inertial reference system are obtained by solving the navigational problem using the Galilean transformation. As the result we have the following equations that are invariant to Galilean transformations:

$$\partial \rho_{(h)}/\partial t + \nabla_{(h)\alpha}\rho_{(h)} v^\alpha = 0 \tag{3.1}$$

$$\rho_{(h)} \frac{dv_\alpha}{dt} + \frac{1}{c^2}\left(\rho_{(h)} U_{(h)} h_{\alpha\beta} - p_{(h)\alpha\beta}\right)\frac{dv^\beta}{dt} = \nabla^\beta_{(h)} p_{(h)\alpha\beta} + Q_{(h)\alpha} \tag{3.2}$$

$$\rho_{(h)} \frac{d}{dt}\left(\frac{v^2}{2} + U_{(h)}\right) + \frac{1}{c^2}\left(\rho_{(h)} U_{(h)} h_{\alpha\beta} - p_{(h)\alpha\beta}\right) v^\alpha \frac{dv^\beta}{dt} = \nabla^\beta_{(h)}\left(p_{(h)\alpha\beta} v^\alpha\right) + \frac{dQ^{(e)}}{dt} + Q_{(h)\alpha} v^\alpha \tag{3.3}$$

where

$$\rho_{(h)} = \rho^\circ \sqrt{\frac{\gamma^\circ}{h^\wedge}}, \quad U_{(h)} = U\big|_{\gamma^\wedge_{\alpha\beta} = h^\wedge_{\alpha\beta}}, \quad Q_{(h)\alpha} = h_{\alpha\beta}\frac{\partial x^\beta}{\partial x^{*\gamma}} Q^{*\gamma}$$

$$h^\wedge_{\alpha\beta} = h_{\gamma\delta}\left(\frac{\partial x^\gamma}{\partial \xi^\alpha}\right)_t \left(\frac{\partial x^\delta}{\partial \xi^\beta}\right)_t, \quad h^\wedge = |h^\wedge_{\alpha\beta}|$$

$$p_{(h)\alpha\beta} = 2\rho_{(h)} \frac{\partial U_{(h)}}{\partial h^\wedge_{\gamma\delta}} \left(\frac{\partial x^\mu}{\partial \xi^\gamma}\right)_t \left(\frac{\partial x^\nu}{\partial \xi^\delta}\right)_t h_{\mu\alpha} h_{\nu\beta}$$

When deriving Eq.(3.3) from (2.11) it is necessary to take into account also formulas (2.9) and (2.10).

If the observer's system of coordinates x^i coincides with the proper system x^{*i}, then on the basis of equality (2.1) we obtain

$$h^\wedge_{\alpha\beta}\big|_{x^i = x^{*i}} = \gamma^\wedge_{\alpha\beta}, \quad \rho_{(h)}\big|_{x^i = x^{*i}} = \rho, \quad U_{(h)}\big|_{x^i = x^{*i}} = U, \quad p_{(h)\alpha\beta}\big|_{x^i = x^{*i}} = p^*_{\alpha\beta}$$

and, consequently, Eqs.(3.1)—(3.3) in the proper reference system are the same as Eqs.(2.9)—(2.11).

Equations (3.1)—(3.3) obviously can be used in Newtonian mechanics for defining the motion of a continuous medium when the condition

$$\varepsilon^2 \equiv \max\left[\frac{v^2}{c^2}, \frac{v x_0}{c^2 t_0}\right] \ll 1$$

where x_0, t_0 are characteristic values of distance and time, is satisfied by the used observer's reference system. In this case the Lorentz transform with the terms $\lesssim \varepsilon^2$ neglected reduces to the Galilean transform and, consequently, Eqs. (3.1)—(3.3) differ from the relativistic equations (2.8) and (2.7) by the terms $\lesssim \varepsilon^2$. The presence in Eqs.(3.2) and (3.3) of terms with coefficients c^{-2} means that for a complete passage to equations of the model of an ideally elastic body of Newtonian mechanics, the kinematic condition that $\varepsilon^2 \ll 1$ is insufficient, and that constraints of dynamic nature which ensure the smallness of the indicated terms are also required. When only condition $\varepsilon^2 \ll 1$ is satisfied the allowance in Eqs.(3.2) and (3.3) for terms with coefficients c^{-2} may prove significant also in Newtonian mechanics, which produces a model of an ideal elastic body with complicated properties owing to internal, i.e. present in the proper reference system, relativistic effects.

Equations (3.2) and (3.3) can also be obtained from the variational equation (1.4) written in three-dimensional form (in an inertial synchronous reference system)

$$\delta \int_{t_1}^{t_2} \int_{V_3} \Lambda_{(h)} dV_3 \, dt + \delta W^* + \delta W = 0 \tag{3.4}$$

$$dV_3 = \sqrt{h} dx^1 dx^2 dx^3 = \sqrt{h^\wedge} d\xi^1 d\xi^2 d\xi^3$$

$$\Lambda_{(h)} = -\rho_{(h)} c^2 - \rho_{(h)} U_{(h)} (S, h_{\alpha\beta}^\wedge, \overset{\circ}{\gamma}{}_{\alpha\beta}^\wedge, K_B^\wedge)$$

$$\delta W^* = \int_{t_1}^{t_2} \int_{V_3} \left[\rho_{(h)} T \delta S + (Q_{(h)\beta} - \rho_{\alpha\beta} a^\alpha) \delta x^\beta + \frac{1}{\sqrt{h^\wedge}} \frac{d}{dt} (\sqrt{h^\wedge} \rho_{\alpha\beta} v^\alpha \delta x^\beta) \right] dV_3 \, dt$$

$$\rho_{\alpha\beta} \equiv \rho_{(h)} \left(1 + \frac{U_{(h)}}{c^2}\right) h_{\alpha\beta} - \frac{1}{c^2} p_{(h)\alpha\beta}, \quad a^\alpha \equiv \frac{dv^\alpha}{dt}, \quad \rho T \frac{dS}{dt} = \frac{dQ^{(e)}}{dt}$$

where variation δ is carried out at constant ξ^α and i.

When $U_{(h)} \equiv 0$ or $c = \infty$, then

$$\rho_{\alpha\beta} = \rho_{(h)} h_{\alpha\beta}$$

and consequently

$$\delta W^* \equiv \int_{t_1}^{t_2} \int_{V_3} [\rho_{(h)} T \delta S + Q_{(h)\beta} \delta x^\beta] \, dV_3 \, dt + \delta \int_{t_1}^{t_2} \int_{V_3} \rho_{(h)} \frac{v^2}{2} dV_3 \, dt$$

When passing to an arbitrarily specified inertial or, generally, deformable reference system for acceleration $a^\alpha \mathbf{э}_\alpha$ defined in an inertial reference system, it is necessary to use the general formula obtained in /5/

$$a^\alpha \mathbf{э}_\alpha = \mathbf{a}_{(r)} + \mathbf{a}_{(t)} + 2 v_{(r)}^\alpha (e_{\alpha\beta} + \omega_{\alpha\beta}) h^{\beta\gamma} \mathbf{э}_\gamma$$

$$e_{\alpha\beta} = \frac{1}{2} (\nabla_{(h)\alpha} v_{(t)\beta} + \nabla_{(h)\beta} v_{(t)\alpha}), \quad \omega_{\alpha\beta} = \frac{1}{2} (\nabla_{(h)\alpha} v_{(t)\beta} - \nabla_{(h)\beta} v_{(t)\alpha})$$

where $\mathbf{v}_{(r)}, \mathbf{a}_{(r)}$ are, respectively, the relative velocity and acceleration in the noninertial reference system, and $\mathbf{v}_{(t)}, \mathbf{a}_{(t)}$ are, respectively, the transfer velocity and acceleration of the considered point of that system.

Equations (3.2) and (3.3) are not of divergent form even in the absence of external forces and heat influx $(Q_{(h)\alpha} = dQ^{(e)} = 0)$, and for $\varepsilon^2 \ll 1$ they differ from the divergent equations (2.7) by the terms $\lesssim \varepsilon^2$. This is explained by the use of Galilean transform for passing from the exact equations (2.10) and (2.11) expressed in the proper reference system to an arbitrary observer's reference system. It may be considered as an indication that the Galilean transform does not exactly reflect the properties of the physical space-time.

The reasoning used above for the model of a perfectly elastic body can be extended to any arbitrary models, since the physical laws formulated in proper reference systems reflect the internal processes intrinsic to particles of continuous medium. These laws are independent of the arbitrary selection of the observer's reference system, and can be considered in the special theory of relativity (STR), as well as in Newtonian mechanics. To obtain respective laws in arbitrarily specified observer's reference systems it is sufficient to solve the navigational problem using specific physical space-time concepts. Thus in STR the Lorentz transform is to be used, while in Newtonian mechanics it is the Galilean transform that has to be used /6/.

The various macroscopic effects of STR that vanish when $c = \infty$ may divided in two types. To the first belong relativistic effects observable in the proper reference system in which a given point of continuous medium at the considered instant of proper time is at rest. Such effects are absent in the case of dust. To the second type belong the additional relativistic effects appearing in STR and are due to passing from the proper reference system at each point of the medium to the observer's reference system that is the same for all points. These are the effects of navigational transformation. Effects of the first type, as in a number of cases also quantum effects (e.g., the phenomenon of ferromagnetism) manifest themselves by their dependence on the determining parameters of the Lagrangian and of functional δW^* in the proper reference systems and can be defined within the scope of Newtonian mechanics. Equations of Newtonian mechanics which take into account only proper relativistic effects in locally determined proper reference systems coincide with the equations of relativistic mechanics when these are expressed in proper reference systems.

The respective global equations of Newtonian mechanics are obtained from equations expressed in proper reference systems after navigational transformation to the observer's global inertial reference system using the Galilean transform, while for obtaining the equation of relativistic mechanics that take into account all relativisitic effects it is necessary to apply the Lorentz transform. The equations of Newtonian mechanics derived in the above manner differ from the usual relativistic equations for the corresponding models of continuous media by terms proportional to c^{-2}. The order of these additonal terms can generally be different and depends on the order of acceleration and of other quantities of thermodynamic nature (formulas (3.2) and (3.3)).

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